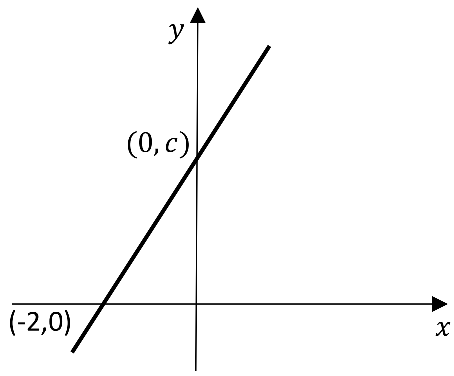
	Prelim Revision 1 Non-Calculator	30
1	Differentiate (a) $g(x) = \ln(3-x)\,(x+5)^3$ (b) $y = \tan^{-1}(3x^2)$	3 3
2	Given that $z = 1 - \sqrt{3}i$, find $\bar{z}$ and express $\bar{z}$ and $\bar{z}^2$ in polar form	4
3	Express $\frac{2x^2+2x-2}{x^3-x}$ in partial fractions Hence give the exact value for $$\int_2^3 \frac{2x^2+2x-2}{x^3-x}\,dx$$ in the form $\ln\left(\frac{m}{n}\right)$	3 3
4	Show that $\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$ Where the integer n is greater than or equal to 3	4
5	Solve $\frac{dy}{dx} = 2x(y+1)$ given that $y = 5$ when $x = 0$	5
6	Write down M_1 the 2×2 matrix associated with an anti-clockwise rotation of $\frac{\pi}{2}$ radians about the origin. Write down M_2 the matrix associated with a reflection in the x-axis Evaluate $M_2 \times M_1$ and describe geometrically the effect of the transformation represented by $M_2 \times M_1$	5

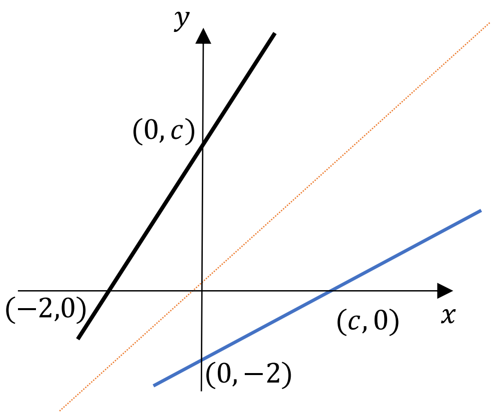
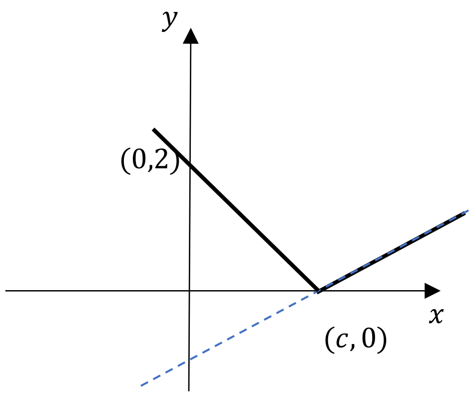
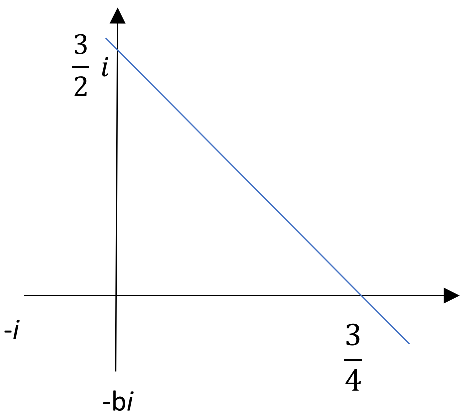
	Prelim Revision 1 Calculator	50
7	The diagram shows part of the graph of a function $f(x)$.  Sketch the graphs for $y = f^{-1}(x)$ and $y = \|f^{-1}(x)\|$ clearly showing the points of intersection with the axes	4
8	Given $\|z-2\| = \|z+i\|$ where $z = x + iy$ Show that $ax + by + c = 0$ for suitable values of a, b and c Indicate on an Argand Diagram the locus of complex numbers z which satisfy $\|z-2\| = \|z+i\|$	3 1
9	A curve is defined by the equations $x = t^2 + t - 1,$ $y = 2t^2 - t + 2$ for all t Show that the point $A(-1,5)$ lies on the curve and obtain an equation for the tangent to the curve at point A	6
10	Use integration by parts to obtain $$\int 6x^2 \cos 2x \; dx$$	5
11	Use the Euclidean Algorithm to find integers x and y such that $$599x + 53y = 1$$	4
12	Prove by contradiction that if x is an irrational number, then $1 + x$ is irrational.	4

13	A curve is defined by $y = \frac{x^5}{(x+1)^4}$, (a) Use logarithmic differentiation to find $\frac{dy}{dx}$ (b) State the equations of any vertical and non-vertical asymptotes for this curve.	 4 3
14	Solve the differential equation $(x+1)\frac{dy}{dx} - 3y = (x+1)^4$ Given that $y = 16$ when $x = 1$, express your answer in the form $y = f(x)$	 6
15	(a) Use the binomial expansion to express z^4 in the form $u + iv$ where u and v are expressions involving $\sin\theta$ and $\cos\theta$ (b) Use de Moivre's theorem write down a second expression for z^4 (c) Using the results from (a) and (b) show that $\frac{\cos 4\theta}{\cos^2\theta} = p\cos^2\theta + q\sec^2\theta + r,$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	 3 1 6

	Prelim Revision 1 – Answers
1a	$$g'(x) = -\frac{1}{3-x} \times (x+5)^3 + \ln(3-x) \times 3(x+5)^2$$ $$= 3\ln(3-x)\,(x+5)^2 - \frac{(x+5)^3}{3-x}$$
1b	$$y = \tan^{-1} u,\ \ u = 3x^2,\quad \frac{dy}{du} = \frac{1}{1+u^2}\quad \frac{du}{dx} = 6x$$ $$\frac{dy}{dx} = \frac{6x}{1+(3x^2)^2} = \frac{6x}{1+9x^4}$$
2	$z = 1-\sqrt{3}i$ so the the conjugate is $\bar{z} = 1+\sqrt{3}i,$ in polar form $\bar{z} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ *and* $\bar{z}^2 = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ *or* $\bar{z}^2 = (1+\sqrt{3}i)(1+\sqrt{3}) = -2+2\sqrt{3}i,$ in polar form $\bar{z}^2 = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
3	Partial Fractions $$\frac{2x^2-2x-2}{x^3-x} = \frac{2x^2-2x-2}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$ $$2x^2-2x-2 = A(x^2-1) + Bx(x-1) + Cx(x+1)$$ if $x = 0$ $\quad -2 = -A,\ \ A = 2$ if $x = 1$ $\quad -2 = 2C,\ \ C = -1$ $\quad$ if $x = -1$ $\quad 2 = 2B,\ \ B = 1$ Hence $$\frac{2x^2-2x-2}{x(x+1)(x-1)} = \frac{2}{x} + \frac{1}{x+1} - \frac{1}{x-1}$$ Integration $$\int_2^3 \frac{2x^2-2x-2}{x^3-x}\,dx = \int_2^3 \frac{2}{x}\,dx + \int_2^3 \frac{1}{x+1}\,dx - \int_2^3 \frac{1}{x-1}\,dx$$ $$= [2\ln x + \ln(x+1) - \ln(x-1)]_2^3$$ $$= \left[\ln\left(\frac{x^2(x+1)}{x-1}\right)\right]_2^3$$ $$= \ln\left(\frac{3^2 \times 4}{2}\right) - \ln\left(\frac{2^2 \times 3}{1}\right) = \ln\left(\frac{36}{24}\right) = \ln\left(\frac{3}{2}\right)$$

4	$$\binom{n+1}{3}-\binom{n}{3}=\frac{(n+1)!}{3!\,(n+1-3)!}-\frac{n!}{3!\,(n-3)!}$$ $$=\frac{(n+1)!}{3!\,(n-2)!}-\frac{n!}{3!\,(n-3)!}$$ $$=\frac{(n+1)!}{3!\,(n-2)!}-\frac{n!\,(n-2)}{3!\,(n-3)!\,(n-2)}$$ $$=\frac{(n+1)!}{3!\,(n-2)!}-\frac{n!\,(n-2)}{3!\,(n-2)!}$$ $$=\frac{n!\,(n+1)-n!\,(n-2)}{3!\,(n-2)!}$$ $$=\frac{n!\,[(n+1)-(n-2)]}{3!\,(n-2)!}$$ $$=\frac{n!\,[3]}{3!\,(n-2)!}$$ $$=\frac{3\times n!}{3\times 2!\,(n-2)!}$$ $$=\frac{n!}{2!\,(n-2)!}=\binom{n}{2}\text{ as required}$$
5	$$\frac{dy}{dx}=2x(y+1)$$ $$\int\frac{1}{y+1}\,dy=\int 2x\,dx$$ $$\ln(y+1)=x^2+C$$

	As an implicit function	As an explicit function
	$$\ln(y+1)=x^2+C$$ $$y=5\text{ when }x=0,$$ $$\ln 6=0+C,$$ $$C=\ln 6$$ $$\ln(y+1)=x^2+\ln 6$$	$$\ln(y+1)=x^2+C$$ $$y+1=e^{x^2+C}$$ $$\boldsymbol{y=Ae^{x^2}-1}\text{ where }\boldsymbol{A=e^c}$$ $$y=5\text{ when }x=0$$ $$5=Ae^0-1,\;5=A-1,\quad A=6$$ $$\boldsymbol{y=6e^{x^2}-1}$$

6	$\boldsymbol{M_1} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\boldsymbol{M_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $M_2 \times M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ this is a reflection in the line $y = -x$
7	The inverse function $y = f^{-1}(x)$ has intercepts at $(c, 0)$ and $(0, -2)$ The modulus function $y = \lvert f^{-1}(x)\rvert$ has intercepts at $(c, 0)$ and $(0,2)$  
8	 $\lvert z-2\rvert = \lvert z-i\rvert$ $\lvert(x-2)+iy\rvert = \lvert x+(y+1)i\rvert$ $\boldsymbol{(x-2)^2 + y^2 = x^2 + (y+1)^2}$ $x^2 - 4x + 4 + y^2 = x^2 + y^2 + 2y + 1$ $-4x + 4 = 2y + 1$ $y = -2x + \frac{3}{2}$

9	At point A $-1 = t^2 + t - 1,$ and $5 = 2t^2 - t + 2$ $0 = t^2 + t,$ and $0 = 2t^2 - t - 3$ $0 = t(t+1)$ $0 = (2t-3)(t+1)$ A common solution to both equations is $t = -1$, thus point A lies on the curve. $\frac{dy}{dt} = 4t - 1$ $\frac{dx}{dt} = 2t + 1$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{4t-1}{2t+1}$ When $t = -1$, $m = 5$ tangent is $y = 5x + 10$
10	$\int 6x^2 \cos 2x \; dx = 6x^2 \times \frac{1}{2}\sin 2x - \int 12x \times \frac{1}{2}\sin 2x \; dx$ $= 3x^2 \sin 2x - 6\int x \sin 2x \; dx$ $= 3x^2 \sin 2x - 6\left[-\frac{1}{2}x\cos 2x - \int 1 \times -\frac{1}{2}\cos 2x \;\; dx\right]$ $= 3x^2 \sin 2x \; + 3x\cos 2x \; - 3\int \cos 2x \;\; dx$ $3x^2 \sin 2x \; + 3x\cos 2x \; - \frac{3}{2}\sin 2x + C$

11	$599 = 53 \times 11 + 16$ $53 = 16 \times 3 + 5$ $16 = 5 \times 3 + 1$ $5 = 1 \times 5 + 0$	$1 = 16 - 3 \times 5$ $1 = 16 - 3 \times (53 - 16 \times 3)$ $1 = -3 \times 53 + 10 \times 16$ $1 = -3 \times 53 + 10 \times (599 - 53 \times 11)$ $1 = 10 \times 599 - 113 \times 53$ $x = 10, \quad y = -113$

12	Assume that $1 + x$ is a rational number and let $1 + x = \frac{a}{b}$ where a and b are integers. Thus $1 + x = \frac{a}{b}$, $x = \frac{a}{b} - 1$, $x = \frac{a-b}{b}$ Since a and b are integers, then it follows that $a - b$ is also an integer and $x = \frac{a-b}{b}$ is a rational number. This is a contradiction.

13	$$\ln y = \ln\left(\frac{x^5}{(x+1)^4}\right)$$ $$\ln y = \ln x^5 - \ln(x+1)^4$$ $$\ln y = 5\ln x - 4\ln(x+1)$$ $$\frac{1}{y}\frac{dy}{dx} = \frac{5}{x} - \frac{4}{x+1}$$ $$\frac{dy}{dx} = y\left(\frac{5}{x} - \frac{4}{x+1}\right)$$ $$\frac{dy}{dx} = \left(\frac{x^5}{(x+1)^4}\right)\left(\frac{x+5}{x(x+1)}\right)$$ $$\frac{dy}{dx} = \frac{x^6+5x^5}{x(x+1)^5}$$ (b) for $y = \frac{x^5}{(x+1)^4}$ there is a vertical asymptote is $x = -1$ Using polynomial long division $y = x - 4 + \frac{10x^3+20x^2+15x+4}{(x+1)^4}$ The non-vertical asymptote is $y = x - 4$
14	$$(x+1)\frac{dy}{dx} - 3y = (x+1)^4 \leftrightarrow \frac{dy}{dx} - \frac{3y}{x+1} = (x+1)^3$$ $$P = -\frac{3}{x+1}, \quad \int P\,dx = -3\ln(x+1), \quad I = e^{-3\ln(x+1)} = \frac{1}{(x+1)^3}$$ $$\frac{1}{(x+1)^3}\frac{dy}{dx} - \frac{1}{(x+1)^3} \times \frac{3y}{x+1} = \frac{1}{(x+1)^3}(x+1)^3$$ $$\int\left(\frac{1}{(x+1)^3}\frac{dy}{dx} - \frac{3y}{(x+1)^4}\right)dx = \int 1dx$$ $$Iy = \int 1\,dx$$ $$\frac{y}{(x+1)^3} = x + C$$ Given that $y = 16$ when $x = 1$ $\quad \frac{16}{2^3} = 1 + C,\ C = 1$ $$\frac{y}{(x+1)^3} = x + 1 \quad \leftrightarrow \quad \boldsymbol{y = (x+1)^4}$$

15	$$z = u + iv = r(\cos\theta + i\sin\theta)$$ $$z^4 = (\cos\theta + i\sin\theta)^4$$ $$= \cos^4\theta + 4\cos^3\theta\,(i\sin\theta) + 6\cos^2\theta\,(i\sin\theta)^2 + 4\cos\theta\,(i\sin\theta)^3 + (i\sin\theta)^4$$ $$= \cos^4\theta + 4\cos^3\theta\sin\theta\,(i) - 6\cos^2\theta\sin^2\theta - 4\cos\theta\sin^3\theta\,(i) + \sin^4\theta$$ $$= (\cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$ Using de Moivre's theorem $z^4 = \cos 4\theta + i\sin\theta$ Given that $\cos 4\theta + i\sin\theta = (\cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$ Using the real part $\cos 4\theta = \cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta$ Divide through by $\cos^2\theta$ $$\frac{\cos 4\theta}{\cos^2\theta} = \frac{\cos^4\theta}{\cos^2\theta} + \frac{\sin^4\theta}{\cos^2\theta} - \frac{6\cos^2\theta\sin^2\theta}{\cos^2\theta}$$ $$\frac{\cos 4\theta}{\cos^2\theta} = \cos^2\theta + \frac{\sin^4\theta}{\cos^2\theta} - 6\sin^2\theta$$ $\sin^2\theta = 1 - \cos^2\theta,$ $\quad \frac{\cos 4\theta}{\cos^2\theta} = \cos^2\theta + \frac{(1-\cos^2\theta)^2}{\cos^2\theta} - 6(1-\cos^2\theta)$ Expand brackets and simplify $$\frac{\cos 4\theta}{\cos^2\theta} = \cos^2\theta + \frac{1 - 2\cos^2\theta + \cos^4\theta}{\cos^2\theta} - 6(1-\cos^2\theta)$$ $$= \cos^2\theta + \frac{1}{\cos^2\theta} - \frac{2\cos^2\theta}{\cos^2\theta} + \frac{\cos^4\theta}{\cos^2\theta} - 6 + 6\cos^2\theta$$ $$= \cos^2\theta + \sec^2\theta - 2 + \cos^2\theta - 6 + 6\cos^2\theta$$ $$= 8\cos^2\theta + \sec^2\theta - 8$$ $$p\cos^2\theta + q\sec^2\theta + r = 8\cos^2\theta + \sec^2\theta - 8$$ $$p = 8, \qquad q = 1, \qquad r = -8$$